UIL MATHEMATICS MAGIC : BOOK I

BASIC MATHEMATICS

1. If 120% of A equals $\frac{2}{5}$ of B and 25% of B equals C, then A is what part of C?

   (A) $\frac{4}{3}$ (B) $\frac{6}{5}$ (C) $\frac{3}{4}$ (D) $\frac{3}{5}$ (E) $\frac{8}{5}$

   $120\% = \frac{6}{5}$ and $25\% = \frac{1}{4}$

   Let $\frac{1}{4}B = C$ and $\frac{6}{5}A = \frac{2}{5}B$; $A = \frac{5}{6}(\frac{2}{5})B = \frac{1}{3}B$

   $$\frac{A}{C} = \frac{\frac{1}{3}B}{\frac{1}{4}B} = \frac{\frac{1}{3}}{\frac{1}{4}} = \frac{1}{3} \div \frac{1}{4} = \frac{1}{3}(\frac{4}{1}) = \frac{4}{3}$$

3. Robin went to the mall for a special sale. She bought 3 sweaters at $19.95 each, 2 pairs of pants at $17.50 each, and a music CD for $8.99. The total cost including tax was $110.85. What was the tax rate?

   (A) 4.2% (B) $6\frac{3}{4}\%$ (C) 10.7% (D) $7\frac{1}{4}\%$ (E) 8.125%

   Let x = the tax rate plus 100% (in decimal form)

   $x[3(19.95) + 2(17.50) + 8.99] = 110.85$
   $x(103.84) = 110.85$
   $x = \frac{110.85}{103.84} = 1.0675$; Thus the tax rate is $.0675 = 6.75\% = 6\frac{3}{4}\%$

20. A stack of quarters has a height of 8 inches. What is the value of a stack of quarters that has a height of 5 feet 8 inches?

   (A) $32.50 (B) $116.00 (C) $232.00 (D) $272.00 (E) $544.00

   Set up a proportion showing the ratio of the number of quarters to the height of the stack. Let Q = number of quarters

$$\frac{Q}{5'8''} = \frac{80}{5} \;;\; \frac{Q}{68} = \frac{80}{5} \;;\; 5Q = 68(80) \;;\; Q = \frac{68(80)}{5} = 1088$$

1088 quarters = 1088($.25) = $272.00

ALGEBRA I

1. Let M and N be the roots of $3x^2 - 5x + 2 = 0$. Find M^2N^2.

(A) $2\frac{7}{9}$ (B) $\frac{4}{9}$ (C) $\frac{4}{5}$ (D) $\frac{5}{9}$ (E) $2\frac{1}{4}$

Note : $M^2N^2 = (MN)^2$, which is asking for the square of the product of the roots. If $Ax^2 + Bx + C = 0$, the product of the roots is $\frac{C}{A}$. The product of the roots of $3x^2 - 5x + 2 = 0$ is $\frac{2}{3}$ and the square of the product of the roots is $\frac{4}{9}$.

2. Simplify : $[b + \frac{b}{a}] \div [b - \frac{b}{a}]$

(A) a - b (B) $\frac{a+1}{a-1}$ (C) a + b (D) $\frac{a+b}{b-a}$ (E) $\frac{1+b}{1-1}$

I suggest that you rewrite the problem as a fraction :

$\dfrac{b+\frac{b}{a}}{b-\frac{b}{a}}$ then multiply the numerator and the denominator by a.

$$\left(\frac{b+\frac{b}{a}}{b-\frac{b}{a}}\right)\left(\frac{a}{a}\right) = \frac{ab+b}{ab-b} = \frac{b(a+1)}{b(a-1)} = \frac{a+1}{a-1}$$

3. A freight train left Harlingen at 5 a.m. traveling 30 mph. At 7 a.m. a passenger train traveling at 50 mph left the same station. What is the distance from the station when the passenger train overtakes the freight train ?

(A) 88 miles (B) 100 miles (C) 150 miles

(D) 180 miles (E) 240 miles

Let t = time it will take the passenger train to overtake the freight train.
Since Distance = Rate x Time (D = RT), then 30(t + 2) = D and 50t = D.
30(t + 2) = 50t ; 30t + 60 = 50t ; 60 = 20t ; t = 3

Since it will take the passenger train 3 hours to overtake the freight train and the distanced traveled by the passenger train is 50t, then 50(3) = 150 miles is the desired distance.

18. Mr. Snail can mow the lawn in 30 minutes using a push mower. Mr. Cutter can mow the same lawn in 18 minutes using a gas mower. How long would it take if both work together?

(A) 6 min (B) $8\frac{8}{9}$ min (C) 11.25 min

(D) $16\frac{2}{3}$ min (E) 24 min

Keep in mind the following : $\frac{Both}{(A)alone} + \frac{Both}{(B)alone} = 1$

$\frac{x}{30} + \frac{x}{18} = 1$; $18(30)(\frac{x}{30} + \frac{x}{18}) = 18(30)(1)$; $18x + 30x = 540$;

$48x = 540$; $x = \frac{540}{48} = 11.25$

38.Line *m* is perpendicular to line *n*. The point of intersection is (- 1, - 3). The point (- 4, 3) lies on line *m*. Which one of the following is a point that lies on line *n*?

(A) (6, 1) (B) (1, - 7) (C) (3, - 2) (D) (5, 0) (E) (- 1, - 1)

Since both (- 1, - 3) and (- 4, 3) lie on line *m* you should find its slope.

Slope of line m = $\frac{-3-3}{-1-(-4)}$ = $\frac{-6}{3}$ = - 2. Since line n is perpendicular to line m its slope is $\frac{1}{2}$. Since (- 1, - 3) lies on line n use it and each of the possible solutions to determine which gives a slope of $\frac{1}{2}$ for line n.

If you use (5, 0) and (- 1, - 3) the slope = $\frac{0-(-3)}{5-(-1)}$ = $\frac{3}{6}$ = $\frac{1}{2}$. Therefore (5, 0) is on line n.

GEOMETRY

1. The point of concurrency of the perpendicular bisectors of the sides of a triangle is the __________.

   (A) circumcenter (B) centroid (C) origin

   (D) orthocenter (E) incenter

   circumcenter : the point of concurrency of the perpendicular bisectors of the sides of a triangle
   centroid : the point of intersection of the medians (sometimes called the median point)
   orthocenter : the point of intersection of the altitudes of the triangle
   incenter : the center of the circle inscribed in a triangle. The point where the angle bisectors of a triangle bisect each other

15. A circle is inscribed in an equilateral triangle with side lengths of 6 units. Find the diameter of the circle.

(A) $\frac{\sqrt{3}}{2}$ (B) $2\sqrt{3}$ (C) $\sqrt{6}$ (D) $3\sqrt{2}$ (E) $\frac{\sqrt{6}}{3}$

Sketch a circle inscribed in an equilateral triangle. Keep in mind that the sides of the triangle are tangent to the circle, which means that they are perpendicular to a radius drawn to the tangent. This radius also bisects the side of the equilateral triangle. Draw a segment from the center to a vertex of the triangle and a radius to one of the sides. This will form a 30°-60°-90° triangle. The long leg will be 3 units long. The short leg is the radius.

long leg = short leg x $\sqrt{3}$
3 = radius x $\sqrt{3}$

radius = $\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \text{ x } \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$

diameter = $2\sqrt{3}$

25. Farmer Frank's right cylindrical water tank is full. The tank is 5 feet high and has a diameter of 8 feet. How many gallons of water is in the tank? (nearest gallon)

(A) 1,880 (B) 1,005 (C) 1,520 (D) 2,507 (E) 1,088

Volume of a right cylinder = $\pi r^2 h$; 1 gallon = 231 cubic inches
Height of tank = 5(12) = 60 inches ; Radius of tank = 4(12) = 48 inches

Volume of tank = $\pi(48)^2(60)$; To determine the number of gallons in the tank, divide the volume (in cubic inches) by 231.

$$\frac{\pi(48)^2(60)}{231} = 1880$$

35. Forester Fred plants a young 4 inch diameter tree. He drives two support poles into the ground on opposite sides of the trees to attach lines for support. The poles enter the soil 18 inches from the base of the tree and stick out of ground 3 feet. The poles lean 30° from vertical away from the tree. What is the distance between the tops of the poles?

Make a sketch of the situation. You will end up with a geometric figure that looks like a trapezoid with the bottom base 40" and the top base x + 40" + x. The trapezoid is isosceles with the legs representing the two support poles. If you draw the altitudes of the support poles perpendicular to the top base you will have two right triangles formed with a 30° angle (x will be the length of the short leg of the right triangle). The length of the longer base represents the distance between the poles which is x + 40" + x.

In a 30°-60°-90° triangle, the hypotenuse is equal to twice the short leg.

Hypotenuse = 2(short leg) ; 36 = 2(x) ; x = 18

Distance between top of poles = 18 + 40 + 18 = 76 ; 76" = 6' 4"

38. Secant AB and tangent BC intersect and point P lies on secant AB. Points A, C. and P lie on the circle and point B is in the exterior of the circle. Find x, If AP = x, BP = 3, and BC = 6.

(A) $\frac{1}{2}$ (B) 2 (C) $3\sqrt{2}$ (D) 5 (E) 9

Sketch a circle. From a point B in the exterior of the circle draw a tangent segment with point C as the point of tangency. From point B draw a secant segment with endpoints, B and A. Point P is on the circle between point A and point B.

Keep in mind the following theorem : If a tangent segment and a secant segment are drawn from the same point outside a circle, the tangent is the geometric mean between the external segment of the secant and the secant.

$\frac{3}{6} = \frac{6}{x+3}$; $3(x + 3) = 6(6)$; $3x + 9 = 36$; $3x = 27$; $x = 9$

ALGEBRA II

5. If $\log_3(x + 3) + \log_3(x) = \log_3(28)$, then x equals __________.

(A) 3 (B) 4 (C) 7 (D) 12 (E) 21

You should be familiar with the following "Laws of Logarithems"

$\log_b M + \log_b N = \log_b MN$
$\log_b M - \log_b N = \log_b \frac{M}{N}$
$\log_b M^p = p\log M$
If $\log_b M = \log_b N$, then $M = N$

$\log_3(x + 3) + \log_3(x) = \log_3(28)$,
$\log_3(x + 3)(x) = \log_3(28)$
$(x + 3)(x) = 28$, Note : Instead of solving the quadratic equation that results, think of two numbers that differ by 3 and whose product is 28. These two numbers are 4 and 7, therefore $x = 4$.

19. One of the factors of $x^4 - 13x^2 + 36$ is :

(A) $x - 9$ (B) $x - 4$ (C) $x + 18$ (D) $x + 3$ (E) $x - 12$

The easiest way to determine which of the listed factors is a factor of $x^4 - 13x^2 + 36$ is to use the Remainder Theorem.

$(9)^4 - 13(9)^2 + 36 = 5625$ (This indicates that |x| must be less than 9)

$(4)^4 - 13(4)^2 + 36 = 84$

$(-3)^4 - 13(-3)^2 + 36 = 0$ (This indicates that $x + 3$ is the factor of $x^4 - 13x^2 + 36$.)

20. P, Q, and R are the real roots of $x^3 - 2x^2 - 5x + 6 = 0$. Find $P^2QR + PQ^2R + PQR^2$.

(A) - 5 (C) - 10 (C) - 12 (D) 8 (E) 30

$P^2QR + PQ^2R + PQR^2 = PQR(P + Q + R)$ Note : They want the product of the product of the roots and the sum of the roots of the equation.

If $ax^3 + bx^2 + cx + d = 0$, the product of the roots is $-\frac{d}{a}$ and the sum of the roots is $-\frac{b}{a}$. In this problem they are looking for $(-\frac{d}{a})(-\frac{b}{a})$ or $\frac{bd}{a^2}$.

$$\frac{bd}{a^2} = \frac{(-2)(6)}{1^2} = -12$$

39. The graph on the x-y plane of $9x^2 + 36x + 4y^2 + 24y + 36 = 0$ lies in quadrant(s) :

(A) III (B) II (C) III & IV (D) II & III (E) I, II, III, & IV

$9x^2 + 36x + 4y^2 + 24y + 36 = 0$
$9(x^2 + 4x + 4) + 4(y^2 + 6y + 9) = -36 + 36 + 36$
$9(x + 2)^2 + 4(y + 3)^2 = 36$
$$\frac{(x+2)^2}{4} + \frac{(y+3)^2}{9} = 1$$

The center of the ellipse is (- 2, - 3)

Two of the vertices are 2 units to the left and right of the center [(0, -3) and (- 4, -3)]. Two of the vertices are 3 units above and below the center [(- 2, 0) and (-2, - 6)]. If you look at the vertices you will notice that two of them are in the third quadrant, one is on the y-axis and the other on the x-axis. Therefore the ellipse lies in Quadrant III.

ADVANCED MATHEMATICS

1. Willy Washer is standing 23.5 m from the base of a building. He sees the top of the building at an angle of elevation of 47°. He sees the bottom of the window at an angle of elevation of 34°. How far is the bottom of the window from the top of the building? (nearest tenth)

   (A) 4.0 m (B) 6.5 m (C) 5.4 m (D) 3.5m (E) 9.3 m

   Make a drawing with Willy at point A, the base of the building at B, the base of the window at point C and the top of the building at point D.

   $\text{Tan } 47° = \frac{DB}{23.5}$; $DB = 23.5\text{Tan}47°$

   $\text{Tan}34° = \frac{CB}{23.5}$; $CB = 23.5\text{Tan}34°$

   DC (the distance from the bottom of the window to the top of the building) = DB - CD = 23Tan47° - 23.5Tan34° = 9.3

6. Find the area, in square units, of the figure bounded by $y = 2 - x^2$ and $y = -x$.

   (A) $2\frac{1}{6}$ (B) $2\frac{5}{6}$ (C) $3\frac{1}{2}$ (D) $4\frac{1}{6}$ (E) $4\frac{1}{2}$

   To find the area bounded by the two functions you will need to use the following formula : $\int_a^b [f(x) - g(x)]dx$.

   Find where the two functions intersect. $2 - x^2 = -x$; $-x^2 + x + 2 = 0$;

   $x^2 - x - 2 = 0$; $(x - 2)(x + 1) = 0$; $x = 2$ and $x = -1$.

   Area = $\int_{-1}^{2} [(2 - x^2) - (-x)]dx = \int_{-1}^{2} [-x^2 + x + 2]dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\,]_{-1}^{2}$

   $[-\frac{8}{3} + 2 + 4] - [\frac{1}{3} + \frac{1}{2} - 2] = 4\frac{1}{2}$

8. Which of the following numbers is considered to be a "deficient" number ?

(A) 6 (B) 20 (C) 24 (D) 28 (E) 46

Suppose you take a positive integer n and add its positive divisors. For example, if n = 18, then the sum is 1 + 2 + 3 + 6 + 9 + 18 = 39. In general, when we do this with n one of the following three things happens

The sum is less than 2n which is called a deficient number (1, 2, 3, 4, 5, 8, 9….), the sum is equal to 2n which is a perfect number (6, 28, 496, …) or the sum is greater than 2n which is called an abundant number (12,18,20, 24, 30,…). There are infinitely many deficient numbers. For example, pK, with p any prime and k >0, is deficient. Also if n is any perfect number, and d is greater than 2n divides n (where 1 < d < n), then d is deficient.

Deficient numbers are sometimes called defective numbers. Primes, prime powers and any divisors of a perfect or deficient number are all deficient. The first few deficient numbers are 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, …

If n = 6, then the sum of the divisors is 1 + 2 + 3 + 6 = 12, therefore 6 is a perfect number. If n = 20, then the sum of the divisors is 1 + 2 + 4 + 5 + 10 + 20 = 42, therefore 20 is an abundant number. If n = 24, then the sum of the divisors is 1 + 2 + 3 + 4 + 6 + 8 + 12 + 24 = 60, therefore 24 is an abundant number. If n = 28, then the sum of the divisors is 1 + 2 + 4 + 7 + 14 + 28 = 56, therefore 24 is an abundant number. If n = 46, then the sum of the divisors is 1 + 2 + 23 + 46 = 72 and 46 is a deficient number.

Note : The sum of the divisors of 46 is 72. If n = 46 and 2n = 92, then the sum of the divisors is less than 2n or 92 which means that 46 is a deficient number.

34. Simplify : $(\csc\theta - \sec\theta)\sin\theta\cos\theta$

(A) $-2\sin\theta\cos\theta$ (B) $\cos\theta + \sin\theta$ (C) $\sin\theta - \cos\theta$

(D) $\cos\theta - \sin\theta$ (E) $2\sin\theta\cos\theta$

$(\csc\theta - \sec\theta)\sin\theta\cos\theta$

$$\left(\frac{1}{\sin\theta} - \frac{1}{\cos\theta}\right)\sin\theta\cos\theta$$

$$\left(\frac{\cos\theta}{\sin\theta\cos\theta} - \frac{\sin\theta}{\sin\theta\cos\theta}\right)\sin\theta\cos\theta$$

$(\frac{\cos\theta - \sin\theta}{\sin\theta\cos\theta})\sin\theta\cos\theta$

$\cos\theta$ - $\sin\theta$

48.If two dice are tossed, what is the probability that the sum of the faces is a prime number.

(A) $\frac{5}{36}$ (B) $\frac{7}{18}$ (C) $\frac{2}{9}$ (D) $\frac{5}{12}$ (E) $\frac{7}{36}$

The largest sum possible when tossing a pair of dice is 12. The only prime numbers less than 12 are 2, 3, 5, 7 and 11.

Sum	Number of outcomes	Sum	Number of outcomes
2	1	7	6
3	2	11	2
5	4		

Probability $= \frac{favorable}{(total)outcomes} = \frac{1+2+4+6+2}{36} = \frac{15}{36} = \frac{5}{12}$

86. The Moby Dick Whaler is traveling on a calm day at 20 km/h on a course of 30° east of north. Captain Ahab measures the wind velocity to be 10 km/h from the west. What is the speed of the actual (resultant) wind velocity? (rounded to the nearest tenth)

(A) 15.0 km/h (B) 25.0 km/h (C) 26.5 km/n

(D) 35.5 km/h (E) 40.5 km/h

Draw an x-y plane. Assume Whaler leaves from the origin in a direction which is 30° east of north (If you draw a segment representing the path of the Whaler it should be going 30° to the right of the positive y-axis). Since the wind is blowing from the west you can draw a segment from the endpoint of the other segment (not the origin) which is parallel to the x-axis. The measure of the angle formed by the two segments drawn is 120°. Draw the segment that represents the actual path of the Whater (resultant).

Use the Law of Cosines ($c^2 = a^2 + b^2 - 2ab\cos C$) to find the resultant.

$x^2 = 20^2 + 10^2 - 2(20)(10)\cos 120°$

$$x = \sqrt{20^2 + 10^2 - 2(20)(10)\cos 120^\circ} = 26.5$$